The topological conjugacy of Cantor minimal systems is not Borel

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Complexity of equivalence relations

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Assume X, Y are Polish spaces (= completely metrizable, separable topological spaces).

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$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

If $E \leq F$ and $F \leq E$, we say that they are *bireducible*.

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If $E \leq F$ and $F \leq E$, we say that they are *bireducible*. A great amount of work has been done to study how various equivalence relations compare against each other in the partial order $\leq$.

Equivalence relations from group actions

Suppose a group G acts on X, it induces the orbit equivalence relation

$$xE_X^G x' \Leftrightarrow \exists g \in G\colon x' = gx.$$

"Isomorphism of ..." as an equivalence relation

Let X, Y be compact metric spaces. Write $X \cong Y$ if they are homeomorphic.
$\cong$ is an equivalence relation on $\{\text{compact metric spaces}\}$.
We need $\cong$ to be an eq. rel. on a Polish space.

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Solution:

- every compact metric space is homeomorphic to a closed subset of a Hilbert cube $H := [0,1]^{\mathbb{N}}$,
- let $K(H) := \{\text{closed subsets of } H\}$. There is a reasonable Polish topology on $K(H)$ (Vietoris topology, given by Hausdorff metric).
- So $\cong$ is a equivalence relation on $K(H)$.

Some types of equivalence relations relevant to this talk

- $S_\infty-$group actions,
- E_{S_∞} which is the largest among those (e.g. isomorphism of countable graphs)
- Polish group actions,
- E_{Polish} which is largest among those (e.g. homeomorphism of compact metric spaces),
- Borel equivalence relations.

Measurable dynamics

A *measure preserving system* (MPS) is a triple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a standard Borel space with a probability measure, and $T\colon X \to X$ is an invertible measurable map s.t.

$$\mu(A) = \mu(T^{-1}A) \quad \text{for all } A \in \mathcal{B}.$$

A MPS is *ergodic* if $\mu(A \triangle T^{-1}A) = 0$ implies $\mu(A) \in \{0, 1\}$.

- Isomorphism of MPS $\geq E_{S_\infty}$ (Hjorth 2001)
- Isomorphism of ergodic MPS is not reducible to any S_∞ action (Hjorth 2001)
- Isomorphism of ergodic MPS is not Borel (Foreman, Rudolph, Weiss 2011)

Topological dynamics

A *topological dynamical system* (TDS for short) is a pair (X, T) where X is a compact metric space and $T\colon X \to X$ is a homeomorphism.
Two TDS (X, T) and (Y, S) are *topologically conjugate* (also called *isomorphic*) if there exists a homeomorphism $\phi\colon X \to Y$ such that $\phi \circ T \circ \phi^{-1} = S$.

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A TDS (X, T) is *minimal* if it has no nontrivial subsystems, i.e.

$$A \subseteq X \text{ closed and } T(A) = A \ \Rightarrow\ A = \emptyset \text{ or } A = X.$$

Equivalently, (X, T) is minimal if $\forall x \in X$ the orbit $\mathcal{O}(x) := \{\, T^n x\colon n \geq 0\}$ is dense in X.

Topological dynamics

A *Cantor system* is a TDS where X is homeomorphic to the Cantor set.
The full shift over $\{0, 1, \dots n-1\}$ is the system $(\{0, 1, \dots n-1\}^{\mathbb{Z}}, \sigma)$, where $\sigma(x)(i) = x(i+1)$.
A *subshift* is a subsystem of full shift (over some n).

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	all	minimal
arbitrary TDS	$\sim E_{Polish}$	not reducible to E_{S_∞} (Peng)
Cantor TDS	$\sim E_{S_\infty}$ (Gao)	$\geq =^+$ (Kaya), not Borel (DGKKK)
subshifts	$\sim E_\infty$ (Clemens)	$\geq E_0$

Isomorphism of Cantor TDS

The systems we just constructed are not minimal.
Gao asked about complexity of $\cong$ of minimal Cantor TDS.
Kaya (2015) proved that $\cong$ of minimal Cantor TDS $\geq =^+$.

Theorem (D, Garcia-Ramos, Kasprzak, Kunde, Kwietniak)

Isomorphism of minimal Cantor TDS is not Borel.

Flip conjugacy

Two minimal Cantor TDS T, S are *flip-conjugate* if $T \cong S$ or $T \cong S^{-1}$.
Given T, we define the *topological full group* $[[T]]$ as follows:
a function $\phi \in \mathrm{Homeo}(C)$ is in $[[T]]$ iff there exists a clopen partition $C = A_1 \sqcup \cdots \sqcup A_n$ and integers $k_1 \dots k_n$ such that $\phi|_{A_i} = T^{k_i}|_{A_i}$.

- $[[T]]$ is countable
- $T \cong_{flip} S$ iff $[[T]]$ and $[[S]]$ isomorphic (Giordano, Putnam, Skau '99)
- $[[T]]$ is amenable (Juschenko, Monod '12)
- Commutator subgroup $D([[T]])$ is simple (Matui '06)
- $T \cong_{flip} S$ iff $D([[T]])$ and $D([[S]])$ isomorphic
- $[[T]]$ is finitely generated iff $(X, T) \cong$ to a subshift

Theorem (D, Garcia-Ramos, Kasprzak, Kunde, Kwietniak)

Flip conjugacy of minimal Cantor TDS is not Borel.

Some ideas about the proof

$C :=$ Cantor set.
$\mathrm{Homeo}(C) := \{f \colon C \to C \mid f \text{ homeomorphism}\}$
$f \cong g$ iff $(C, f) \cong (C, g)$.

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Another viewpoint:
write $\sigma\colon C^{\mathbb{Z}} \to C^{\mathbb{Z}}$ for the map $\sigma(x)(n) = x(n+1)$.
If $A \subseteq C^{\mathbb{Z}}$ is closed, perfect, and $\sigma A = A$, then $(A, \sigma|_A)$ is a Cantor TDS.
Write $\mathcal{K}^p_\sigma(C^{\mathbb{Z}})$ for the family of all such sets.
Every Cantor TDS can be realized in this manner: A Cantor TDS (C, f) is isomorphic to $(A, \sigma|_A)$, where

$$A = \{...f^{-1}x, x, fx, f^2x, \ldots |x \in C\}.$$

For all our purposes, these two viewpoints are equivalent.

A rooted countable tree (vertices might have infinite degree) can be viewed as a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that if $w \in T$ then all prefixes of w are in T.
A tree is *ill-founded* if it has an infinite branch.
$IF := \{T \in 2^{\mathbb{N}^{<\mathbb{N}}} : T \text{ ill-founded}\}$ is complete analytic subset of *Trees*.
We will build a Borel reduction

$$Trees \ni T \mapsto (X_T, X'_T) \in \mathcal{K}^p_\sigma(C^{\mathbb{Z}}) \times \mathcal{K}^p_\sigma(C^{\mathbb{Z}})$$

such that $X_T \cong X'_T$ iff T is ill-founded. This will be a Borel reduction from IF to $\cong$, which implies $\cong$ is not Borel.

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If (X_n, σ) are subshifts, and $\pi_n\colon X_{n+1} \to X_n$ is a sequence of factor maps, we define their inverse limit $\varprojlim(X_n, \pi_n) := (X, \sigma)$, where

$$X = \{(x_1, x_2, x_3 \dots)\colon x_n \in X_n \text{ and } \pi_n(x_{n+1}) = x_n\},$$

$$\sigma\colon (x_1, x_2, x_3 \dots) \mapsto (\sigma x_1, \sigma x_2, \sigma x_3 \dots).$$

The result **does** depend on the factor maps π_n.
On the other hand, factor maps are far from unique: if $\pi\colon (X, T) \to (Y, S)$ is a factor map, then so is $\psi \circ \pi$, where $\psi \in \mathrm{Aut}(Y, S)$.

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Idea: take a sequence of subshifts $(X_n, \sigma)_{n\geq 1}$ and factor maps $(\pi_n)_{n\geq 1}$. Then take different factor maps $(\pi'_n)_{n\geq 1}$, where $\pi'_n = \psi_n \pi_n$ for some $\psi_n \in \mathrm{Aut}(X_n, \sigma)$. Then take two inverse limits X, X'. Can we find a reasonable condition for these two inverse limits to be isomorphic?

Definition

Sequence of subshifts (X_n, σ) *and factor maps* $\pi_n \colon X_{n+1} \to X_n$ *is* ***blended*** *if* $\forall i \geq j \geq 1$ *every factor map* $\zeta \colon X_i \to X_j$ *can be written as* $\psi \circ \pi_{i-1} \circ \cdots \circ \pi_j$, *where* $\psi \in Aut(Y, S)$.

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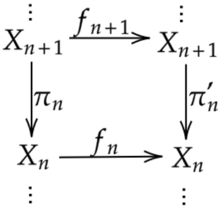
Lemma (2)

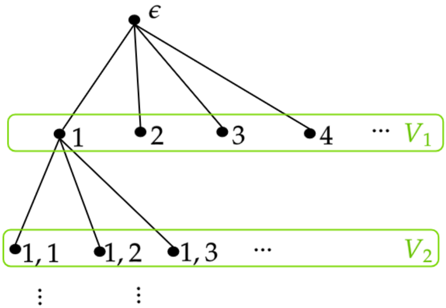
X_n, π_n as above. Let $\pi'_n \colon X_{n+1} \to X_n$ be a different sequence of factor maps. X, X' resulting inverse limits. TFAE:

(i) (X, σ) and (X', σ) isomorphic,

(ii) there exist automorphisms $f_n \in Aut(X_n)$ such that $\pi'_n f_{n+1} = f_n \pi_n$.



To a tree T, one can attach an inverse system of groups [FRW].



$V_n^T :=$ vertices of T at level n,
$G_n^T :=$ the vector space over $\mathbb{F}_2$ with basis V_n,
$\rho_n^T \colon G_{n+1}^T \to G_n^T$ defined by values on generators: $\rho_n^T(v) = \mathrm{parent}(v)$.

Lemma (3)

Let T be a tree. It's possible to find minimal subshifts (X_n, σ) and factor maps $\pi_n\colon X_{n+1} \to X_n$ such that

- $Aut(X_n, \sigma) = G_n^T \times \mathbb{Z}$, *where $\mathbb{Z}$ corresponds to the shift map*
- *Given two maps $F \in Aut(X_{n+1}, \sigma)$, $f \in Aut(X_n, \sigma)$, we have $\pi_n F = f\pi_n$ iff $f = \rho_n^T(F)$.*
- *the collection $(X_n, \sigma)_{n\geq 1}$, $(\pi_n)_{n\geq 1}$ is blended.*

Combining everything together, we get

Lemma (4)

Let

$$\mathcal{Y} := \{T, (g_n)_{n\geq 1} \colon T \in Trees, g_n \in G_n^T \text{ for all } n\}.$$

Then we have a Borel map

$$\Phi \colon \mathcal{Y} \ni (T, (g_n)_{n\geq 1}) \mapsto (X, X') \in \mathcal{K}_\sigma^p(C^{\mathbb{Z}}) \times \mathcal{K}_\sigma^p(C^{\mathbb{Z}})$$

such that TFAE:

(i) X, X' *isomorphic,*

(ii) there exist a sequence $f_n \in G_n^T$ *such that* $g_n + \rho_n(f_{n+1}) = f_n$

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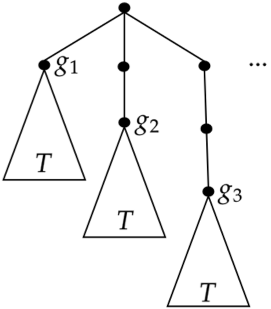
Proof.

Pick X_n, π_n as in previous lemma. Take $X := \varprojlim(X_n, \pi_n)$ and $X' := \varprojlim(X_n, g_n\pi_n)$. Equivalence of conditions (i) and (ii) follow from Lemma 2. □

Given a tree T, define new tree $\Psi(T)$ and $g_n \in G_n^{\Psi(T)}$ to be as in picture.



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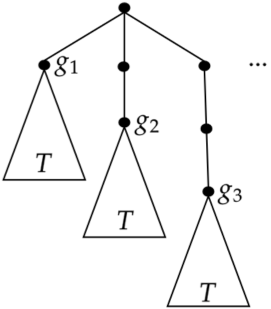

Lemma (5)

TFAE:
(i) There exists a sequence $f_n \in G_n^{\Psi(T)}$ such that $g_n + \rho_n(f_{n+1}) = f_n$,
(ii) T is ill-founded.

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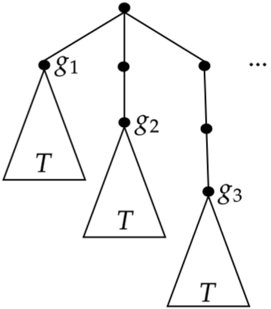

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(ii) T is ill-founded.

Corollary

$$Trees \ni T \xrightarrow{lem.\ 5} (\Psi(T), (g_n)_{n\geq 1}) \xrightarrow{lem.\ 4} X, X'$$

is a reduction from IF to $\cong$ of minimal Cantor systems.